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'$***|**|$' - two bars on top of each other. Which two?

Stars and Bars Poll

Mark whats correct.

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

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Dividing 5 dollars among Alice, Bob and Eve.

Poll

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dis == distinguishiable unique = one ball in each bin.

(A) dis $=>$ n^k
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Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
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□

Pascal's Triangle

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	0	
1		1

Pascal's Triangle

		0		
	1		1	
1		2		1

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

0

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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Foil (4 terms)

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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\end{array}
$$

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1 2 1
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 & & & 1 & & 1 & & & \\
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Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

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element i **is in** or **is not** in the subset: 2 poss.

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Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
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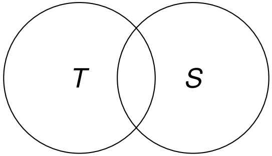

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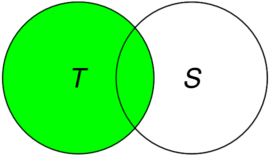

In T. $\implies |T|$

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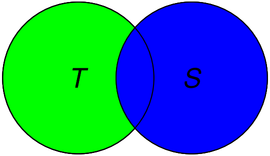

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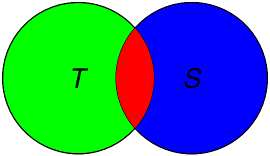

$S \cap T$

In T. $\implies$ $|T|$
In S. $\implies$ $+$ $|S|$
Elements in $S \cap T$ are counted twice.

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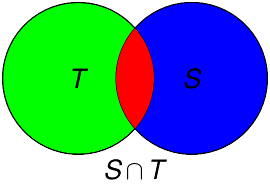

In T. $\implies$ $|T|$
In S. $\implies$ $+\ |S|$
Elements in $S \cap T$ are counted twice.
Subtract. $\implies$ $-|S \cap T|$

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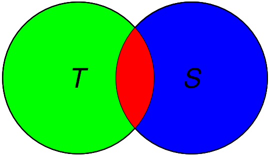

$S \cap T$

In T. $\implies |T|$

In S. $\implies + \; |S|$

Elements in $S \cap T$ are counted twice.

Subtract. $\implies -|S \cap T|$

$$|S \cup T| = |S| + |T| - |S \cap T|$$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$

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T = phone numbers with 7 as second digit. $|T| = 10^9$.

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Inclusion/Exclusion

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$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.

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$(x+y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m$.

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Proof: m factors in product: $(x+y)(x+y)\cdots(x+y)$.

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Get a term $x^{m-i} y^i$ by choosing i factors to use for y.

are $\binom{m}{i}$ ways to choose factors where y is provided. $\square$

For $x = 1, y = -1$,

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Idea: how many times is each element counted?

Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

Counted $\binom{m}{i}$ times in ith summation.

Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1}\binom{m}{m}$.

Binomial Theorem:
$(x+y)^m = \binom{m}{0}x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + \cdots \binom{m}{m}y^m$.

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$0 = (1-1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m\binom{m}{m}$

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$\implies \quad 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \cdots + (-1)^{m-1}\binom{m}{m}$.

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$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Each element counted once!

Summary.

First Rule of counting:

Summary.

First Rule of counting: Objects from a sequence of choices:

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n_i possibilitities for ith choice :

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First Rule of counting: Objects from a sequence of choices:
n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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Second Rule of counting:

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n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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Summary.

First Rule of counting: Objects from a sequence of choices:

n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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Count with order:

Summary.

First Rule of counting: Objects from a sequence of choices:
n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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Count with order: Divide number of orderings. Typically: $\binom{n}{k}$.

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Add number of each subtract intersection of sets.

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LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.

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$\binom{n}{k}$ counts subsets of $n+1$ items without first item.

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$\binom{n}{k}$ counts subsets of $n+1$ items without first item.
Disjoint

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First Rule of counting: Objects from a sequence of choices:
n_i possibilititities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.
$\binom{n}{k}$ counts subsets of $n+1$ items without first item.
Disjoint – so add!

Midterm Review

Now...

First there was logic...

A statement is true or false.

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A statement is true or false.

Statements?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

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Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$?

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

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$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

Example: $x = 3$

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Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

First there was logic...

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Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$?

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

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Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

$x = y$?

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$3 = 4 - 1$? Statement!

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$n > 3$? Predicate: $P(n)$!

$x = y$? Predicate: $P(x, y)$!

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$3 = 4 - 1$? Statement!

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Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

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$x + y$?

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$x = y$? Predicate: $P(x, y)$!

$x + y$? No.

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$x + y$? No. An expression, not a statement.

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Quantifiers:

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Quantifiers:

$(\forall x)\ P(x)$.

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$x + y$? No. An expression, not a statement.

Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

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$x + y$? No. An expression, not a statement.

Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

$(\exists x)\ P(x)$.

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$(\forall n \in N), n^2 \geq n$.

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$(\forall x \in R)(\exists y \in R) y > x$.

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Predicate?

$n > 3$? Predicate: $P(n)$!

$x = y$? Predicate: $P(x, y)$!

$x + y$? No. An expression, not a statement.

Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

$(\exists x)\ P(x)$. There exists an x, where $P(x)$ is true.

$(\forall n \in N), n^2 \geq n$.

$(\forall x \in R)(\exists y \in R) y > x$.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

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Propositional Expressions and Logical Equivalence

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
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Proofs: truth table or manipulation of known formulas.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

..and then proofs...

Direct: $P \implies Q$

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even?

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even? $a = 2k$

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$a^2 = 2(2k^2)$

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Integers closed under multiplication!

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Useful for prove something does not exist:

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:

...jumping forward..

Contradiction in induction:
contradict place where induction step doesn't hold.

...jumping forward..

Contradiction in induction:

contradict place where induction step doesn't hold.

Well Ordering Principle.

...jumping forward..

Contradiction in induction:
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Well Ordering Principle.
Stable Marriage:

...jumping forward..

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Well Ordering Principle.
Stable Marriage:
first day where canditate gets worse job on string.

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first day where any job is rejected by optimal candidate.

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 Do not exist.

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    first day where canditate gets worse job on string.
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...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

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$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

...and then induction...

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Induction on n.

...and then induction...

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Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

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$3^{2n+2} - 1 =$

...and then induction...

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Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

$$3^{2n+2}-1 = 9(3^{2n})-1$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$ (by induction hypothesis)

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

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Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \end{aligned}$$

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Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

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Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$
\begin{aligned}
3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\
&= 9(8d+1) - 1 \\
&= 72d + 8
\end{aligned}
$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d + 1) - 1 \\ &= 72d + 8 \\ &= 8(9d + 1) \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8|3^{2n}-1$.

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$$\begin{aligned} 3^{2n+2}-1 &= 9(3^{2n})-1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1)-1 \\ &= 72d+8 \\ &= 8(9d+1) \end{aligned}$$

Divisible by 8.

...and then induction...

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Divisible by 8.

□

Stable Matching: a study in definitions and WOP.

n-jobs, n-candidate.

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Pairing.
Set of pairs (m_i, w_j) containing all entities *exactly* once.

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How many pairs?

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How many pairs? *n*.

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Entities in pair are **partners** in pairing.

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Rogue Couple in a pairing.
A m_j and w_k who like each other more than their partners

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Stable Pairing.
Pairing with no rogue couples.

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Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

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Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

No, for roommates problem.

TMA.

Job Propose or reject Matching Algorithm:

TMA.

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Each Day:

TMA.

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All jobs propose to favorite non-rejecting candidate.

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Useful Algorithmic Definitions:

TMA.

Job Propose or reject Matching Algorithm:

Each Day:
All jobs propose to favorite non-rejecting candidate.
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Useful Algorithmic Definitions:
Job **crosses off** candidate who rejected him.

TMA.

Job Propose or reject Matching Algorithm:

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Connected Graph: one connected component.

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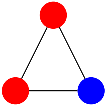

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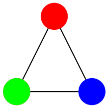

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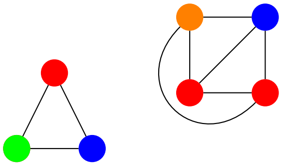

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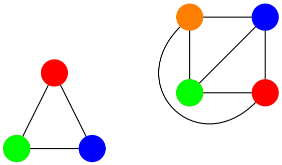

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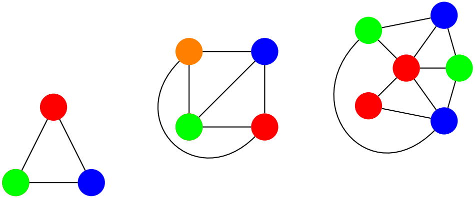

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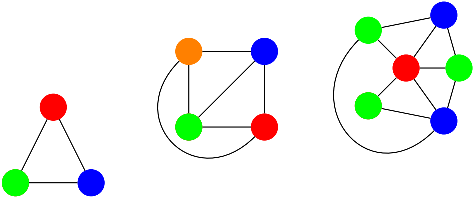

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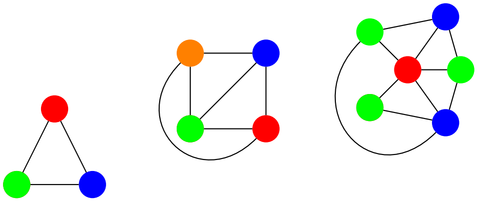

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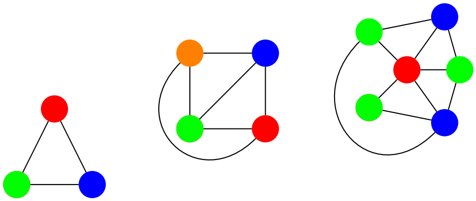

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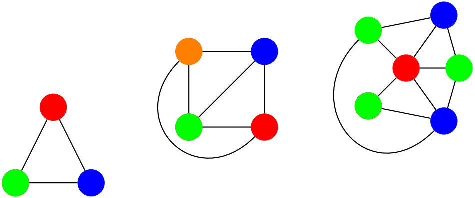

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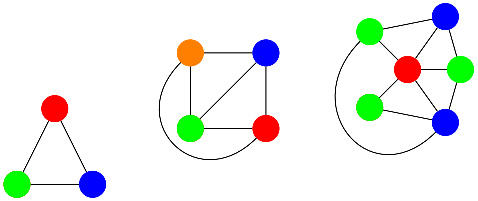

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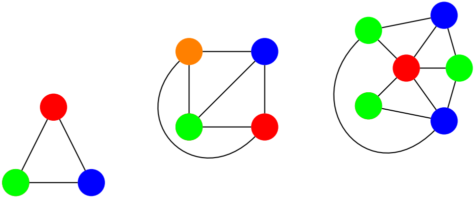

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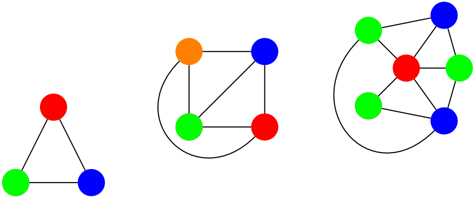

Notice that the last one, has one three colors.
  Fewer colors than number of vertices.
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Interesting things to do.

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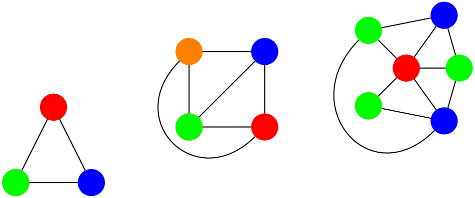

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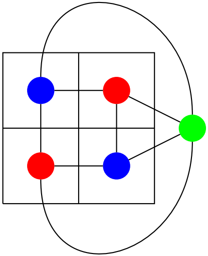
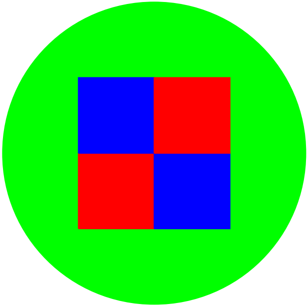
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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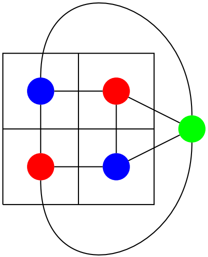

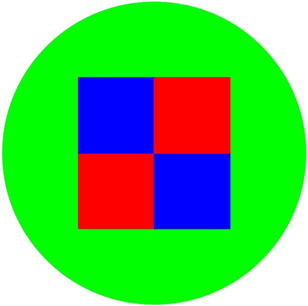

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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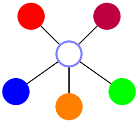

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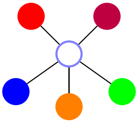

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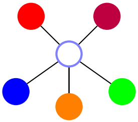

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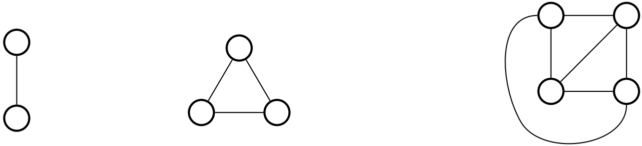
One will work.

Graph Types: Complete Graph.



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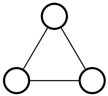

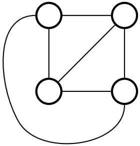

K_n, $|V| = n$

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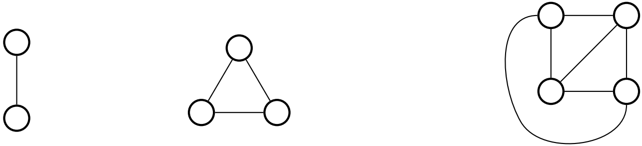

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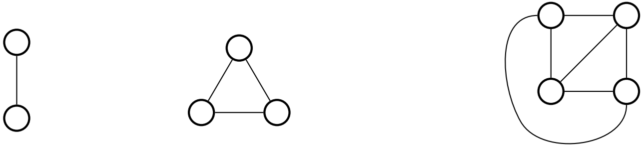

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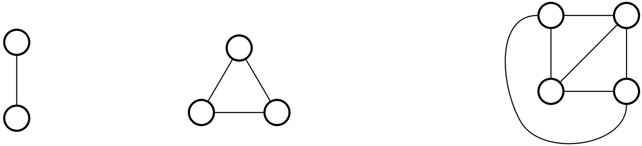

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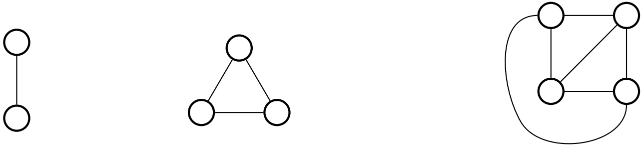

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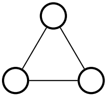

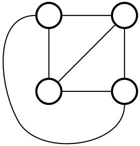

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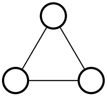

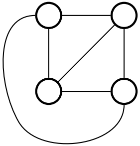

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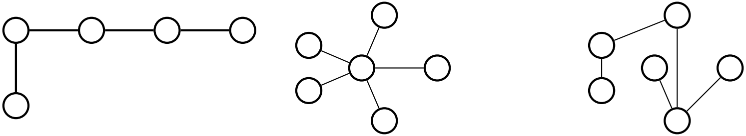
Very connected.
Lots of edges: $n(n-1)/2$.

Trees.



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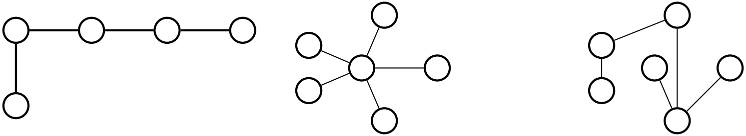

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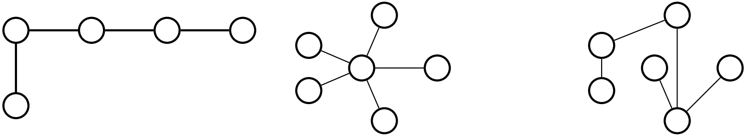

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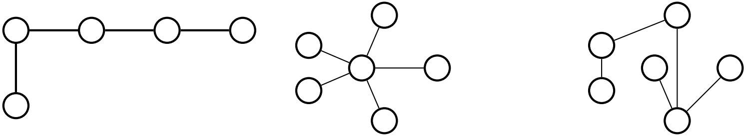

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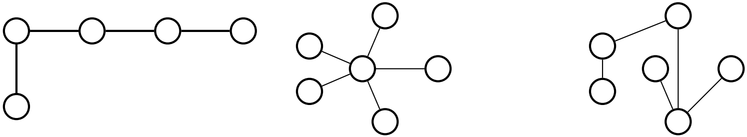

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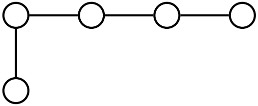

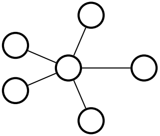

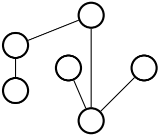

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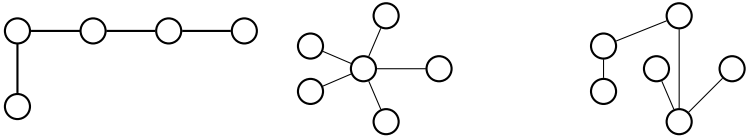

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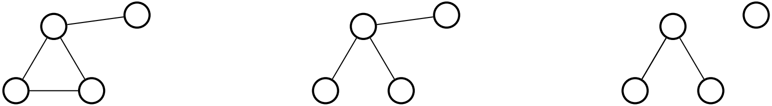

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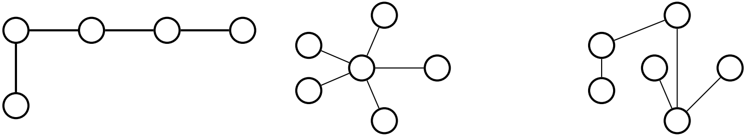

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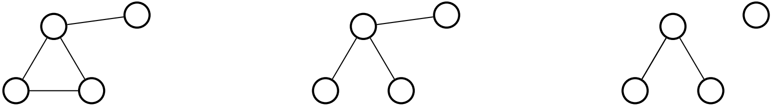

Minimally connected, minimum number of edges to connect.

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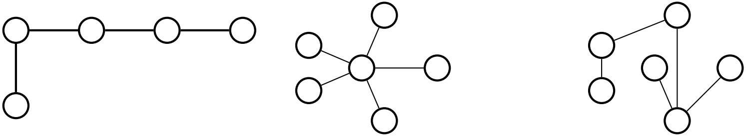

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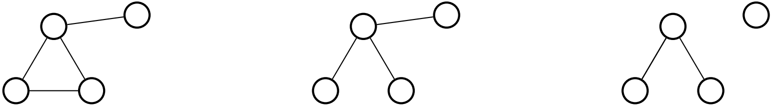

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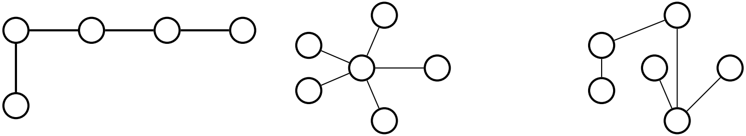

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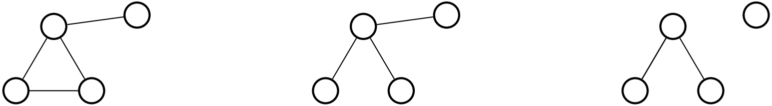

Minimally connected, minimum number of edges to connect.

Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

Hypercubes.

Hypercube

Hypercubes. Really connected.

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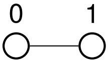

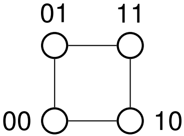

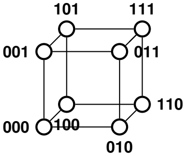

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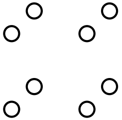

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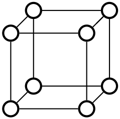

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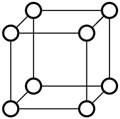

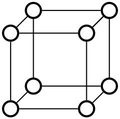

Recursive Definition.

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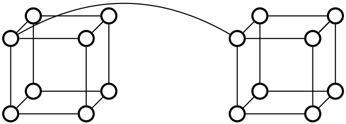

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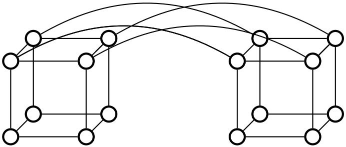

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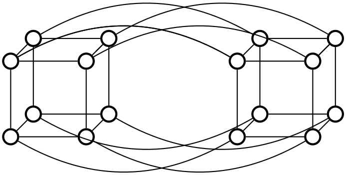

Hypercube:properties

Rudrata Cycle: cycle that visits every node.

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Good communication network!

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Arithmetic modulo m.

Elements of equivalence classes of integers.

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Additive inverses are intuitively negative numbers.

Modular Arithmetic and multiplicative inverses.

$3^{-1} \pmod 7$?

Modular Arithmetic and multiplicative inverses.

$3^{-1} \pmod{7}$? 5

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x has inverse modulo m if and only if $gcd(x,m) = 1$.

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$d = e^{-1} = -17 = 43 = \pmod{60}$

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Verify: Check $C = E(C)$.

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$D(E(m,K),k) = (m^e)^d \mod N = m$.

Signature scheme:

$S(C) = D(C)$.
Announce $(C, S(C))$

Verify: Check $C = E(C)$.

$E(D(C,k),K) = (C^d)^e = C \pmod{N}$

Modular Arithmetic in a minute.

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Product of elts == for range/domain: a^{p-1} factor in range.

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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Roots fact: Any degree $\leq d$ polynomial has at most d roots.

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(Within 1 bit of) b-bits are missing: one $P(i)$.

Within 1 of optimal number of bits.

Summary. Error Correction.

Communicate n packets, with k erasures.

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How many packets?

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How many packets? $n+k$

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How to encode? With polynomial, $P(x)$.

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Of degree?

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Recover?

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Nonlinear equations.

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Reed-Solomon codes.

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Reed-Solomon codes. Welsh-Berlekamp Decoding.

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Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!

Midterm format

Time: 120 minutes.

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Time: 120 minutes.

Some short answers.

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
   Know material well:

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
   Know material well: fast,

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.

Midterm format

Time: 120 minutes.

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  Get at ideas that you learned.
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   Know material medium:

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
    Know material well: fast, correct.
    Know material medium: slower,

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.
Know material medium: slower, less correct.

Midterm format

Time: 120 minutes.

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Get at ideas that you learned.

Know material well: fast, correct.

Know material medium: slower, less correct.

Know material not so well:

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well:	fast, correct.
Know material medium:	slower, less correct.
Know material not so well:	Uh oh.

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Some longer questions.

Midterm format

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Some longer questions.
Proofs,

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Proofs, algorithms,

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Proofs, algorithms, properties.
Not so much calculation.

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Wrapup.

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Other issues....

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  sp21@eecs70.org
  Private message on piazza.

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