Due: Friday, 03/12 at 10:00 PM

Grace period until Friday, 03/12 at 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) Computer programs that halt. *Hint: How can we represent a computer program?*

(f) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{A}$.

(g) The set of infinite-length strings over the English alphabet.

2 Countability Proof Practice

(a) A disk is a 2D region of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \le r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap. Is this set always countable, or potentially uncountable?

(*Hint*: Attempt to relate it to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$)

(b) A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap. Is this set always countable, or potentially uncountable?

(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)

(c) Is the set containing all increasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \geq f(y)$) countable or uncountable? Prove your answer.

(d) Is the set containing all decreasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \leq f(y)$) countable or uncountable? Prove your answer.

3 Finite and Infinite Graphs

The graph material that we learned in lecture still applies if the set of vertices of a graph is infinite. We thus make a distinction between finite and infinite graphs: a graph $G = (V, E)$ is finite if V and E are both finite. Otherwise, the graph is infnite. As examples, consider the graphs

- $G_1 = (V = \mathbb{Z},\ E = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid |i - j| = 1\})$

- $G_2 = (V = \mathbb{Z},\ E = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i < j\})$

- $G_3 = (V = \mathbb{Z}^2,\ E = \{((i, j), (k, l)) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \mid (i = k \wedge |j - l| = 1) \vee (j = l \wedge |i - k| = 1)\})$

Observe that G_1 is a line of integers, G_2 is a complete graph over all integers, and G_3 is an grid of integers. Prove whether the following sets of graphs are countable or uncountable

(a) The set of all finite graphs $G = (V, E)$, for $V \subseteq \mathbb{N}$

(b) The set of all infinite graphs over a fixed, countably infinite set of vertices (in other words, they all have the same vertex set).

(c) The set of all graphs over a fixed, countably infinite set of vertices, the degree of each vertex is exactly two. For instance, every vertex in G_1 (defined above) has degree 2.

(d) We say that graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if the exists some bijection $f : V \to V'$ such that $(u, v) \in V$ iff $(f(u), f(v)) \in V'$. Such a bijection f is called a **graph isomorphism**. Suppose we consider two graphs to be the equivalent if they are isomorphic. The idea is that if we relabel the vertices of a graph, it is still the same graph. Using this definition of "being the same graph", can you conclude that the set of trees over countably infinite vertices is countable?

(*Hint:* Begin by showing that for any graph isomorphism f, and any vertex v, $f(v)$ and v have the same degree)

4 Unions and Intersections

Given:

- A is a countable, non-empty set. For all $i \in A$, S_i is an uncountable set.

- B is an uncountable set. For all $i \in B$, Q_i is a countable set.

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $A \cap B$

(b) $A \cup B$

(c) $\bigcup_{i \in A} S_i$

(d) $\bigcap_{i \in A} S_i$

(e) $\bigcup_{i \in B} Q_i$

(f) $\bigcap_{i \in B} Q_i$

5 Unprogrammable Programs

Prove whether the programs described below can exist or not.

(a) A program $P(F, x, y)$ that returns true if the program F outputs y when given x as input (i.e. $F(x) = y$) and false otherwise.

(b) A program P that takes two programs F and G as arguments, and returns true if F and G halt on the same set of inputs (or false otherwise).

6 Computability

Decide whether the following statements are true or false. Please justify your answers.

(a) The problem of determining whether a program halts in time 2^{n^2} on an input of size n is undecidable.

(b) There is no computer program `Line` which takes a program P, an input x, and a line number L, and determines whether the Lth line of code is executed when the program P is run on the input x.

7 Computations on Programs

(a) Is it possible to write a program that takes a natural number n as input, and finds the shortest arithmetic formula which computes n? For the purpose of this question, a formula is a sequence consisting of some valid combination of (decimal) digits, standard binary operators ($+$, $\times$, the "^" operator that raises to a power), and parentheses. We define the length of a formula as the number of characters in the formula. Specifically, each operator, decimal digit, or parentheses counts as one character.

(*Hint:* Think about whether it's possible to enumerate the set of possible arithmetic formulas. How would you know when to stop?)

(b) Now say you wish to write a program that, given a natural number input n, finds another program (e.g. in Java or C) which prints out n. The discovered program should have the minimum execution-time-plus-length of all the programs that print n. Execution time is measured by the number of CPU instructions executed, while "length" is the number of characters in the source code. Can this be done?

(*Hint:* Is it possible to tell whether a program halts on a given input within t steps? What can you say about the execution-time-plus-length of the program if you know that it does not halt within t steps?)

8 Kolmogorov Complexity

Compressing a bit string x of length n can be interpreted as the task of creating a program of fewer than n bits that returns x. The Kolmogorov complexity of a string $K(x)$ is the length of an optimally-compressed copy of x; that is, $K(x)$ is the length of shortest program that returns x.

(a) Explain why the notion of the "smallest positive integer that cannot be defined in under 280 characters" is paradoxical.

(b) Prove that for any length n, there is at least one string of bits that cannot be compressed to less than n bits.

(c) Say you have a program K that outputs the Kolmogorov complexity of any input string. Under the assumption that you can use such a program K as a subroutine, design another program P that takes an integer n as input, and outputs the length-n binary string with the highest Kolmogorov complexity. If there is more than one string with the highest complexity, output the one that comes first lexicographically.

(d) Let's say you compile the program P you just wrote and get an m bit executable, for some $m \in \mathbb{N}$ (i.e. the program P can be represented in m bits). Prove that the program P (and consequently the program K) cannot exist.

(*Hint*: Consider what happens when P is given a very large input n.)