CS 70 Spring 2021 — Discrete Mathematics and Probability Theory

HW 3

Due: Friday, 02/12 at 10:00 PM
Grace period until Friday, 02/12 at 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

2 Touring Hypercube

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- v_0 and v_k are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if n is even. (*Hint: Euler's theorem*)

(b) Show that every hypercube has a Hamiltonian tour.

3 Connectivity

Consider the following claims regarding connectivity:

(a) Prove: If G is a graph with n vertices such that for any two non-adjacent vertices u and v, it holds that $\deg u + \deg v \geq n-1$, then G is connected.

[*Hint:* Show something more specific: for any two non-adjacent vertices u and v, there must be a vertex w such that u and v are both adjacent to w.]

(b) Give an example to show that if the condition $\deg u + \deg v \geq n-1$ is replaced with $\deg u + \deg v \geq n-2$, then G is not necessarily connected.

(c) Prove: For a graph G with n vertices, if the degree of each vertex is at least $n/2$, then G is connected.

(d) Prove: If there are exactly two vertices with odd degrees in a graph, then they must be in the same connected component (meaning, there is a path connecting these two vertices).

[*Hint:* Proof by contradiction.]

4 Hamiltonian cycle in a dense graph.

A *Hamiltonian cycle* is a cycle where each vertex appears exactly once.

We will show that a simple graph with $n \geq 3$ vertices and where every vertex has degree strictly greater than $n/2$ has an Hamiltonian cycle.

We will do a proof by induction, but it is a bit different from what we are used to.

The idea is to say there is no smallest counterexample. That is, we assume we have a graph $G = (V, E)$ where adding any edge produces a graph with a Hamiltonian cycle.

Consider two vertices u and v that are non-adjacent and thus adding edge (u, v) implies that $G + (u, v)$ has a Hamiltonian cycle. Now G does not have any Hamiltonian cycles so every Hamiltonian cycle in $G + (u, v)$ uses edge (u, v).

This implies that there is a path, $u, v_1, \ldots, v_{n-2}, v$, in G that visits every vertex once and whose endpoints are u and v.

Argue that G has a Hamiltonian cycle. This contradicts the assumption that G did not have a Hamlitonian cycle and completes a proof of the claim since this means there is no smallest counterexample to the claim.

(Hint: Use the fact the u and v have high degree to argue that there is v_i on the path $v_2, \ldots, v_{n-2}$ where $(u, v_i) \in E$ and $(v, v_{i-1}) \in E$.)

5 Modular Practice

(a) Calculate $72^{316} \bmod 7$.

(b) Solve the following system for x:

$$\begin{aligned} 3x &\equiv 4+y && \pmod 5 \\ 2(x-1) &\equiv 2y && \pmod 5 \end{aligned}$$

(c) If it exists, find the multiplicative inverse of 31 mod 23 and 23 mod 31.

(d) Let n, x be positive integers. Prove that x has a multiplicative inverse modulo n if and only if $\gcd(n,x) = 1$. (Hint: Remember an iff needs to be proven both directions. The gcd cannot be 0 or negative.)

6 Modular Inverses

Recall the definition of inverses from lecture: let $a, m \in \mathbb{Z}$ and $m > 0$; if $x \in \mathbb{Z}$ satisfies $ax \equiv 1 \pmod m$, then we say x is an **inverse of** a **modulo** m.

Now, we will investigate the existence and uniqueness of inverses.

(a) Is 3 an inverse of 5 modulo 10?

(b) Is 3 an inverse of 5 modulo 14?

(c) Is each $3+14n$ where $n \in \mathbb{Z}$ an inverse of 5 modulo 14?

(d) Does 4 have inverse modulo 8?

(e) Suppose $x, x' \in \mathbb{Z}$ are both inverses of a modulo m. Is it possible that $x \not\equiv x' \pmod m$?

7 Just Can't Wait

Joel lives in Berkeley. He mainly commutes by public transport, i.e., bus and BART. He hates waiting while transferring, and he usually plans his trip so that he can get on his next vehicle immediately after he gets off the previous one (zero transfer time, i.e. if he gets off his previous vehicle at 7:00am he gets on his next vehicle at 7:00am). Tomorrow, Joel needs to take an AC Transit bus from his home stop to the Downtown Berkeley BART station, then take BART into San Francisco.

(a) The bus arrives at Joel's home stop every 22 minutes from 6:05am onwards, and it takes 10 minutes to get to the Downtown Berkeley BART station. The train arrives at the station every 8 minutes from 4:25am onwards. What time is the earliest bus he can take to be able to transfer to the train immediately? Show your work. (Find the answer without listing all the schedules. Hint: derive an equation relating the bus number and train number and then work in modular arithmetic to get rid of one of the variables to give a set of possible train numbers.)

(b) Joel has to take a Muni bus after he gets off the train in San Francisco. The commute time on BART is 33 minutes, and the Muni bus arrives at the San Francisco BART station every 17 minutes from 7:12am onwards. What time is the earliest bus he could take from Berkeley to ensure zero transfer time for both transfers? If all bus/BART services stop just before midnight, is it the only bus he can take that day? Show your work.