1 Countability and the Halting Problem

Prove the Halting Problem using the set of all programs and inputs.

a) What is a reasonable representation for a computer program? Using this definition, show that the set of all programs are countable. *(Hint: Python Code)*

b) We consider only finite-length inputs. Show that the set of all inputs are countable.

c) Assume that you have a program that tells you whether or not a given program halts on a specific input. Since the set of all programs and the set of all inputs are countable, we can enumerate them and construct the following table.

	x_1	x_2	x_3	x_4	$\dots$
p_1	H	L	H	L	$\dots$
p_2	L	L	L	H	$\dots$
p_3	H	L	H	L	$\dots$
p_4	L	H	L	L	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

An H (resp. L) in the ith row and jth column means that program p_i halts (resp. loops) on input x_j. Now write a program that is not within the set of programs in the table above.

d) Find a contradiction in part a and part c to show that the halting problem can't be solved.

2 Fixed Points

Consider the problem of determining if a function F has any fixed points. That is, given a function F that takes inputs from some (possibly infinite) set $\mathscr{X}$, we want to know if there is any input $x \in \mathscr{X}$ such that $F(x)$ outputs x. Prove that this problem is undecidable.

3 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program P prints "Hello World!" before running the kth line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.