Poll: How big is infinity?

Poll: How big is infinity?

Mark what's true.

(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.

Same Size. Poll.

Two sets are the same size?

Same Size. Poll.

Two sets are the same size?

(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.

Same Size. Poll.

Two sets are the same size?

(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.

(A), (B).

Same Size. Poll.

Two sets are the same size?

(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.

(A), (B).
(C)?

Next up: how big is infinity.

Next up: how big is infinity.

- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

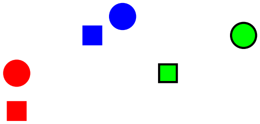
Infinite!

How big are the reals or the integers?

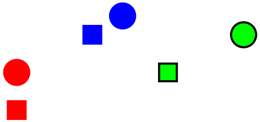
Infinite!

Is one bigger or smaller?

Same size?

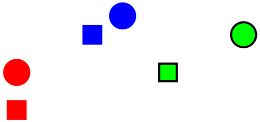


Same size?



Same number?

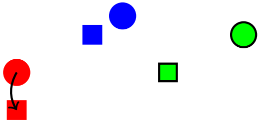
Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

Same size?

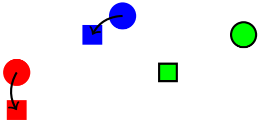


Same number?

Make a function f : Circles $\to$ Squares.

f(red circle) $=$ red square

Same size?



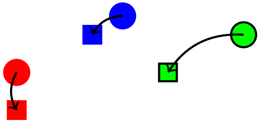
Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

Same size?



Same number?

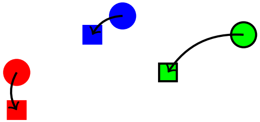
Make a function f : Circles $\to$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

Same size?



Same number?

Make a function f : Circles $\to$ Squares.

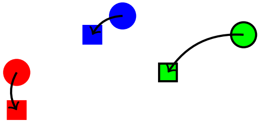
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

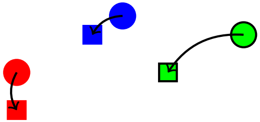
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

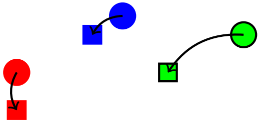
f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

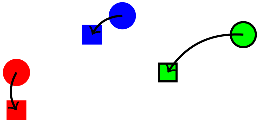
f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Onto.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

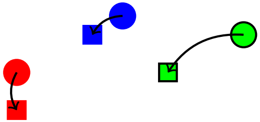
f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

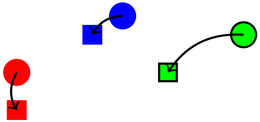
Onto. Each square mapped to from some circle .

Same size?



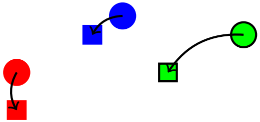
Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Same size?



Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Same size?



Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

One to One:

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Isomorphism principle.

Given a function, $f : D \to R$.

One to One:

For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

or

$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

One to One:

For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

or

$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:

Isomorphism principle.

Given a function, $f : D \to R$.

One to One:

For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

or

$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:

If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

0, 1, 2, 3, ...

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where’s 0?

Which is bigger?

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n,

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$,

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0,1,2,3,\ldots$.

Positive integers. $1,2,3,\ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z)$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n+1) - 1 = n$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto for $\mathbb{N}$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto for $\mathbb{N}$

Bijection!

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n+1) - 1 = n$.
Onto for $\mathbb{N}$

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto for $\mathbb{N}$

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto for $\mathbb{N}$

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where's zero?

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto for $\mathbb{N}$

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where's zero? "Comes from 1."

A bijection is a bijection.

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n + 1$.

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \rightarrow 1,$

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \to 1, 1 \to 2,$

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n + 1$. $0 \rightarrow 1, 1 \rightarrow 2, \ldots$

A bijection is a bijection.

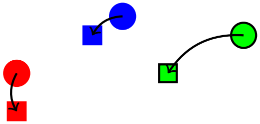
Notice that there is a bijection between N and Z^+ as well.
$f(n) = n + 1$. $0 \to 1, 1 \to 2, \ldots$

Bijection from A to $B \implies$ a bijection from B to A.

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \to 1, 1 \to 2, \ldots$

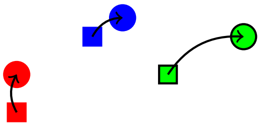
Bijection from A to $B \implies$ a bijection from B to A.



A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \rightarrow 1, 1 \rightarrow 2, \ldots$

Bijection from A to $B \implies$ a bijection from B to A.

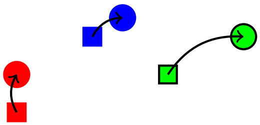


Inverse function!

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \to 1, 1 \to 2, \ldots$

Bijection from A to $B \implies$ a bijection from B to A.



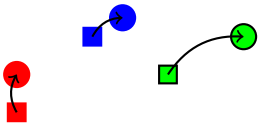
Inverse function!

Can prove equivalence either way.

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \to 1, 1 \to 2, \ldots$

Bijection from A to $B \implies$ a bijection from B to A.



Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto:

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$.

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \to E$.

$f(n) \to 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \to E$.

$f(n) \to 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one:

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y$.

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,

All integers?

What about Integers, Z?
Define $f : N \rightarrow Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2$

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$

All integers?

What about Integers, Z?
Define $f : N \rightarrow Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\ldots.$

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\ldots.$

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\ldots$.

Onto: For any $z \in Z$,

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\dots$.

Onto: For any $z \in Z$,
if $z \geq 0$, $f(2z) = z$ and $2z \in N$.

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\dots$.

Onto: For any $z \in Z$,
if $z \geq 0$, $f(2z) = z$ and $2z \in N$.
if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in N$.

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\ldots$.

Onto: For any $z \in Z$,
if $z \geq 0$, $f(2z) = z$ and $2z \in N$.
if $z < 0$, $f(2|z|-1) = z$ and $2|z|+1 \in N$.

Integers and naturals have same size!

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing "is" a bijection with a subset of natural numbers.

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing "is" a bijection with a subset of natural numbers.
Function $\equiv$ "Position in list."

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing "is" a bijection with a subset of natural numbers.
Function $\equiv$ "Position in list."
If finite: bijection with $\{0, \ldots, |S| - 1\}$

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
$\dots$	$\dots$

Notice that: A listing "is" a bijection with a subset of natural numbers.
Function $\equiv$ "Position in list."
If finite: bijection with $\{0,\dots,|S|-1\}$
If infinite: bijection with N.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

“Output element of S”,
“Output next element of S”

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

“Output element of S”,
“Output next element of S”
...
Any element x of S has *specific, finite* position in list.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0,$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1,$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1,$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"

...

Any element x of S has *specific, finite* position in list.

$Z = \{0, 1, -1, 2,$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2,$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, ...\}$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of *S*",
"Output next element of *S*"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots, \}$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, ...\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of *S*",
"Output next element of *S*"

...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1?

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of *S*",
"Output next element of *S*"
$\ldots$
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of *S*",
"Output next element of *S*"

...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, ...\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of *S*",
"Output next element of *S*"
$\ldots$
Any element *x* of *S* has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

61A

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

61A $\equiv$ streams!

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of *S*",
"Output next element of *S*"
$\ldots$
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

61A $\equiv$ streams! Not Sp20/Fa20.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.

All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi,$

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0,$

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1,$

Enumeration example.

All binary strings.
$B = \{0, 1\}^*$.

$B = \{\phi, 0, 1, 00,$

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11,$

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

$B = \{\phi; , 0, 00, 000, 0000, ...\}$

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

$B = \{\phi; , 0, 00, 000, 0000, ...\}$
Never get to 1.

More fractions?

Enumerate the rational numbers in order...

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:
any two fractions has another fraction between it.

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:
any two fractions has another fraction between it.

Can't even get to "next" fraction!

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:
any two fractions has another fraction between it.

Can't even get to "next" fraction!

Can't list in "order".

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared ???

Pairs of natural numbers.

Enumerate in list:

Pairs of natural numbers.

Enumerate in list:
$(0,0)$,

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),$

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),$

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),$

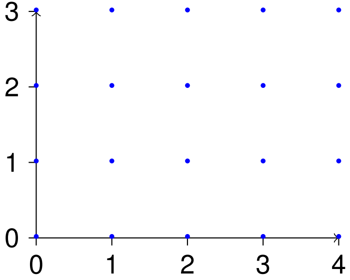
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),$

Pairs of natural numbers.

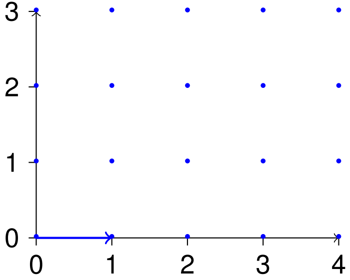
Enumerate in list:

$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



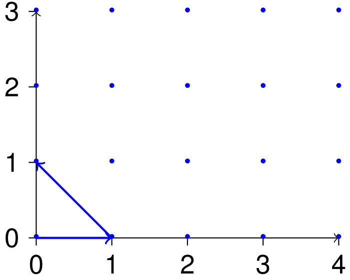
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



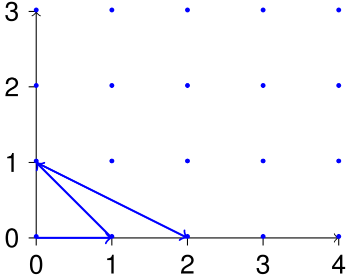
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



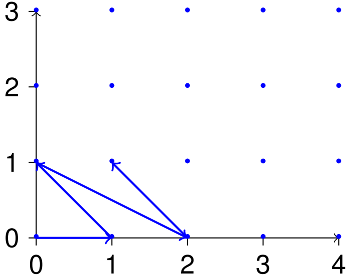
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



Pairs of natural numbers.

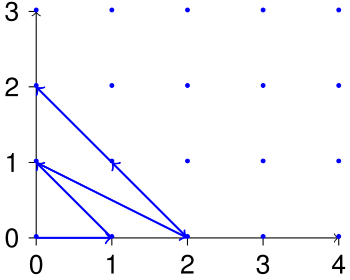
Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



Pairs of natural numbers.

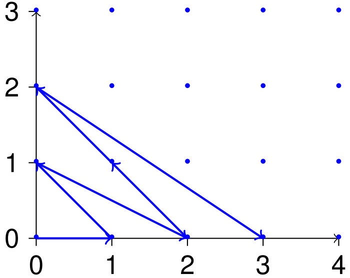
Enumerate in list:

$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



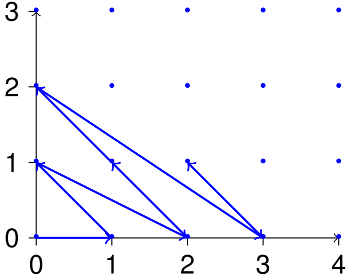
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



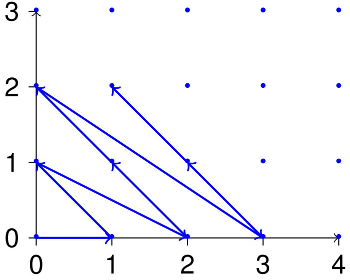
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



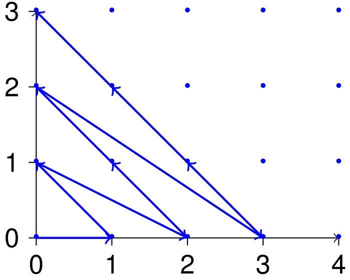
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



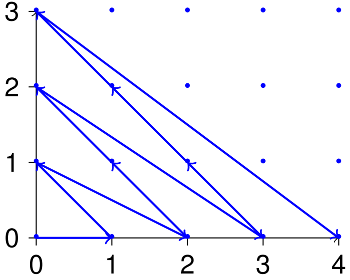
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



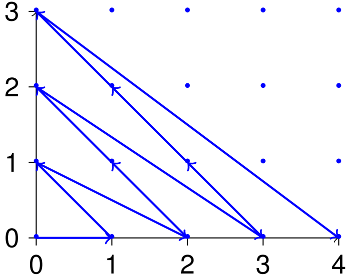
Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



Pairs of natural numbers.

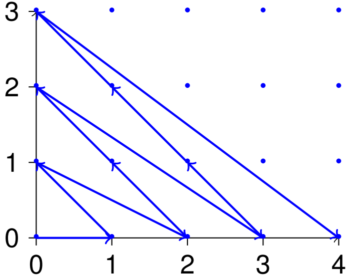
Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



The pair (a,b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!

Pairs of natural numbers.

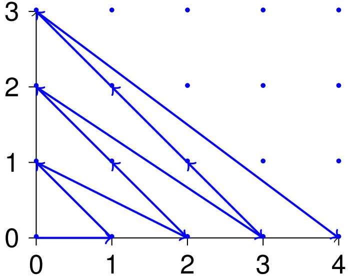
Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



The pair (a,b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!
(i.e., "triangle").

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$

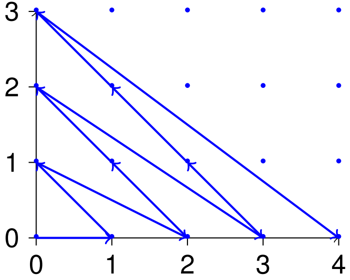


The pair (a,b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!
(i.e., “triangle”).

Countably infinite.

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



The pair (a,b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!
(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

Poll.

Enumeration to get bijection with naturals?

(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.

Poll.

Enumeration to get bijection with naturals?

(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.

(B),(C), (F).

Rationals?

Positive rational number.

Rationals?

Positive rational number.
Lowest terms: a/b

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable.

Rationals?

Positive rational number.
Lowest terms: a/b
$a,b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ???

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.

Rationals?

Positive rational number.
Lowest terms: a/b
$a,b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.
    Interleave Streams in 61A

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.
    Interleave Streams in 61A

The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000...

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162...

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441...

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558...

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312...

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312... Some real number

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312... Some real number

Diagonalization.

If countable, there a listing, L contains all reals.

Diagonalization.

If countable, there a listing, L contains all reals. For example

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number:

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .7

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .77

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct “diagonal” number: .776

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .7767

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .77677

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677 ...

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: $.77677\ldots$

Diagonal Number:

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .77677 ...

Diagonal Number: Digit *i* is 7 if number *i*'s *i*th digit is not 7

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit *i* is 7 if number *i*'s *i*th digit is not 7
and 6 otherwise.

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .77677...

Diagonal Number: Digit *i* is 7 if number *i*'s *i*th digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit *i* is 7 if number *i*'s *i*th digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.
5. Show that t is in S.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, *L*, that contains all subsets of *N*.

Define a diagonal set, *D*:

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, *L*, that contains all subsets of *N*.

Define a diagonal set, *D*:
If *i*th set in *L* does not contain *i*, $i \in D$.
otherwise $i \notin D$.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, *L*, that contains all subsets of *N*.

Define a diagonal set, *D*:
If *i*th set in *L* does not contain *i*, $i \in D$.
otherwise $i \notin D$.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \not\in D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, *L*, that contains all subsets of *N*.

Define a diagonal set, *D*:
If *i*th set in *L* does not contain *i*, $i \in D$.
otherwise $i \notin D$.

D is different from *i*th set in *L* for every *i*.
$\implies$ *D* is not in the listing.

D is a subset of *N*.

L does not contain all subsets of *N*.

Contradiction.

Theorem: The set of all subsets of *N* is not countable.
(The set of all subsets of *S*, is the **powerset** of *N*.)

Diagonalize Natural Number.

Natural numbers have a listing, L.

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

Differs from all elements of listing.

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

Differs from all elements of listing.

D is a natural number...

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

Differs from all elements of listing.

D is a natural number... Not.

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

Differs from all elements of listing.

D is a natural number... Not.

Any natural number has a finite number of digits.

Diagonalize Natural Number.

Natural numbers have a listing, *L*.

Make a diagonal number, *D*:
differ from *i*th element of *L* in *i*th digit.

Differs from all elements of listing.

D is a natural number... Not.

Any natural number has a finite number of digits.

"Diagonal number construction" requires an infinite number of digits.

Poll: diagonalization Proof.

Mark parts of proof.

Poll: diagonalization Proof.

Mark parts of proof.

(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can't be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.

Poll: diagonalization Proof.

Mark parts of proof.

(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can't be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.

(B), (C)?, (D), (E)

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$
If both in $[0,1/2]$,

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$
If both in $[0,1/2]$, a shift

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

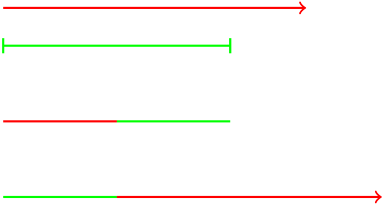
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : R^+ \rightarrow [0, 1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



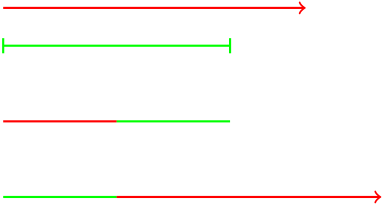
One to one. $x \neq y$
If both in $[0, 1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0, 1/2]$

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



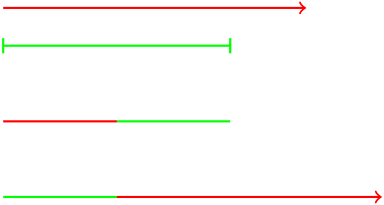
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



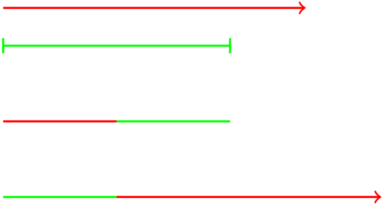
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



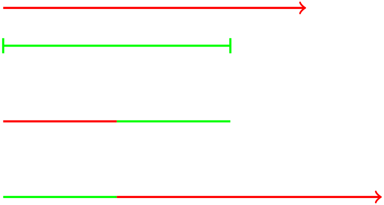
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't,

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



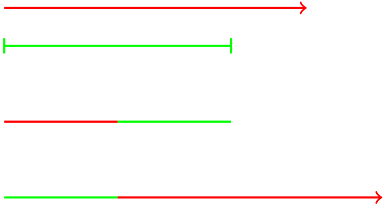
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



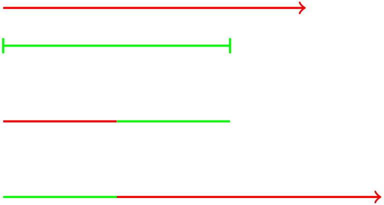
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



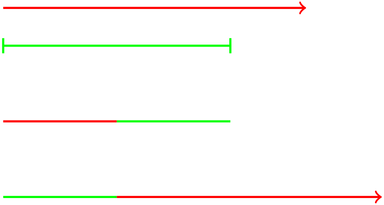
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.
Bijection!

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.
Bijection!

$[0,1]$ is same cardinality as nonnegative reals!

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Resolution of hypothesis?

Resolution of hypothesis?

Gödel. 1940.
Can't use math!

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....