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1. **Pledge.**

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2. **Long Ago. Pts: 2/2/2/2/3/3/3**

1. $A \vee \neg(B \wedge C) \equiv A \vee (\neg B \vee \neg C)$
   Answer: True. $A \vee \neg(B \wedge C) \equiv A \vee (\neg B \vee \neg C)$

2. $\neg \forall x, \exists y, Q(x,y) \equiv \exists x, \exists y, \neg Q(x,y)$
   Answer: False. $\neg \forall x, \exists y, Q(x,y) \equiv \exists x, \forall y, \neg Q(x,y)$

3. $P \implies Q$ is logically equivalent to $\neg Q \vee P$.
   Answer: False. Its $\neg P \vee Q$.

4. Consider a stable matching instance where S is the job optimal stable pairing. Consider a run of the job-propose matching algorithm, where one candidate c rejects a job j that they should not have in one step. That is, c recieves an offer from j and j' and chooses j' instead of j, when j was ahead of j' in their preference list.

   Let P be the resulting pairing.

   (a) For every instance, job j cannot do better in P than in S. (Better means get a partner who they prefer more.)
   Answer: True. If $(j,c) \in S$, of course j does worse since they propose in order. It $(j,c) \notin S$, consider that $(j',c) \in S$ and $(j',c) \in P$ then the pairings are the same overall. We have $(j',c) \notin S$ and $(j',c) \in P$, but then the candidate prefers j to j' and also no others are preferred more, so $(j,c) \in S$. This is a contradiction.

   (b) Every candidate other than c does as well or better in P than in S.
   Answer: False. The job that c takes over j and perhaps ends up with may be first in another candidate's list and have been paired with that candidate in S.

   (c) Every job other than j does as well or better in P than in S.
   Answer: False. Consider that $(j,c), (j'',c'') \in S$, and c'' is first on j'''s list. Moreover, wehn j is rejected by c proposes to c'' who prefers j to j''. Now j'' is out of luck.

3. **Graphs: Pts: 3/3/3/2**

1. Consider a graph on n vertices with exactly one cycle and m edges. What is the number of connected components? (Hint: $m \leq n$.)

Answer: $n - m + 1$. Remove an edge on the cycle and add $n - 1 - (m - 1) = n - m$ edges to connect components and get a tree. Each addition reduced the number of components by 1. Thus the number of components is $n - m + 1$.

2. For a tree on n vertices, what is the expected number of connected components if each edge is deleted with probability $1/3$?

   Answer: $1 + (n - 1)/3$. $E[\text{deleted edges}] = (n - 1)/3$. The number of components is 1 + the number of deleted edges.

3. If we delete every edge with probability $1/2$ from an Eulerian graph on n vertices, what is the expected number of odd degree vertices in the remaining graph?

   Answer: $n/2$. Each vertex is equally likely to be of odd or even degree as at least one edge.

4. Every simple cycle is 2-colorable.

   Answer: False. An odd cycle is not 2 colorable.

4. **Mostly Modular.Pts: 2/2/2/2/3/3/3/3/3/3/3**

   1. $\forall$ nonzero $x, y \in \mathbb{N}$ $gcd(x, y \mod x) = gcd(x, y)$.

      Answer: True. $x = id, y = jd$ iff and only if $x = id$ and $y - kjd = md$ for integers k, m.

   2. $\forall$ nonzero $x, y \in \mathbb{N}$, $gcd(x, x \mod y) = gcd(x, y)$.

      Answer: False. Let $x = 8, y = 3$.

   3. Give an example of positive integers for a and n where

      $$(1 \cdot 2 \cdots (n - 1))a^{n-1} \neq (1 \cdot 2 \cdots (n - 1)) \pmod{n}.$$

      a: *n:*

      Answer: $a = 2, n = 4$. One can check but the key is that a and n are not relatively prime.

   4. Let $S = \{x : x \in \{1, \ldots, 34\}$ and $gcd(x, 35) = 1\}$?

      (a) What is the size of the set S?

         Answer: 24. $(7 - 1)(5 - 1)$ from RSA. Removing multiples of 7 and 5 from the range $\{0, \ldots, 34\}$.

      (b) What is $a^{|S|+1} \pmod{35}$ for $a \in S$?

         Answer: a. $a^{|S|+1} - a = 0 \pmod{35}$ from the proof for RSA.

      (c) What is $a^{|S|+1} \pmod{35}$ for $a \notin S$?

         Answer: a. $a^{|S|+1} - a = 0 \pmod{35}$ from the proof for RSA.

   5. What is $18^{-1} \pmod{13}$?

      Answer: 8. $18 = 5 \pmod{13}$. $8 \times 5 = 40 = 1 \pmod{13}$

   6. If $x = 1 \pmod{13}$ and $x = 0 \pmod{18}$ then what is $x \pmod{234}$? (Note: $234 = 18 \times 13$)

      Answer: 144. Using chinese remainder theorem except on need one term. $1 \times 8 \times 18 \pmod{234}$.

   7. For primes, p and q, where $e = d^{-1} \pmod{(p-1)(q-1)}$?

      (a) What is $a^{ed} \pmod{q}$? (Answer cannot use e or d, but may use numbers, a, p or q.)

         Answer: a. This is the RSA scheme.

      (b) Find an $x \leq pq$, where $p | (a^{ed} - x)$. (Answer is an expression that may use a, p, and q.)

         Answer: a. This is the RSA scheme.

5. **Polynomials.Pts: 3/3/3**

1. Given a polynomial, $x^3 + a_2x^2 + a_1x + a_0$ modulo 7 with roots at 3, 1, and 6. What is a_0? (Notice that the coefficient of x^3 is 1.)

   Answer: 3 (mod 7). The polynomial is $(x-3)(x-1)(x-6)$ so $a_0 = -3 \times 1 \times 6$ (mod 7).

2. Working (mod 5), find a polynomial modulo 5 of degree 2 that has roots at 0 and 3, and goes through point $(2,3)$

   Answer: $(x-0)(x-3)$. Consider $\Delta_2(x) = \frac{(x-0)(x-3)}{-2} = 2(x-0)(x-3)$, multiply it by 3 as the other values are zero.

3. Consider that one encodes a message of n numbers (mod m), by forming a degree $n-1$ polynomial using the numbers as coefficients, and sending $2n-1$ points. If each point is erased with probability $1/2$, what is the probability that the original message can be reconstructed? (Hint: each pattern of erasures is equally probable.)

   Answer: $1/2$. If majority are delivered, this workds. For each configuration of with k erasures there is a configuration with $2n-1-k$ erasaures. In one case, the majority are delivered. Thus, the probability that the majority is delivered is $1/2$.

6. Countability/Computability Pts: 2/2/2/2/2/2

1. For every pair of distinct rational numbers there is a rational number in between them.

   Answer: True. If x, y are rational than so is $x + y/2$.

2. The rational numbers are uncountable.

   Answer: False. One can enumerate the positive ones by enumerating pairs (a,b) of natural numbers in order of the sum of the numerators. Then any rational a/b is represented at least once in this list.

   To get all natural numbers alternate between positive and negative numbers to form a list.

3. There is a program that takes a program P, an input x, and a number n and determines whether P run on input x ever writes to memory location n.

   Answer: False. Assume there is such a program WRITES. Given an instance P and x of the halting problem we modifying P to write into one memory location higher. Then, we can modify it to write to location 0 if it exits. Let P' be the resulting program. Then we can call WRITES on P', x and 0. WRITES will answer affirmatively if and only if P halts on x.

4. There is a program that takes a program P, an input x, and a number n and determines whether P run on input x ever writes to any memory location $i \geq n$.

   Answer: True. One can simulate the program and check if it ever writes to a memory location larger than n. If it runs forever and never writes to a memory location larger than n, then a configuration of the first n memory location and the program counter must repeat. This can be kept track of as well.

5. A program "knows" a real number if it takes an integer n and outputs the nth bit of the real number. (Note: positive values of n signify to the left of the decimal point, and negative ones to the right.)

   (a) There is a program that knows π.

      Answer: True. There are a number of ways to calculate pi: e.g., approximate a circle with a polygon and compute the perimeter.

   (b) For every real number x, there is a program that knows x.

      Answer: False. There are more real numbers(uncountable) than there are programs (countable).

7. A little counting.Pts: 3/3/3/3

1. What is the number of ways to have k strictly positive numbers that add up to n?
   Answer: $\binom{n-1}{k-1}$. Have $n-k$ stars and $k-1$ bars. The ith number correspond to 1+ the number of stars before the ith bar, and the kth number is the stars after the $k-$1th bar.

2. What is the number of ways to produce a sequence of numbers $0 < x_1 < x_2 < \cdots < x_k < n$?
   Answer: $\binom{n-1}{k}$. This is choosing a subset of $n-1$ numbers without replacement.

3. What is the number of ways to produce a sequence of numbers $0 \leq x_1 \leq x_2 \leq \cdots \leq x_k < n$?
   Answer: $\binom{k+n-1}{n-1}$. We have k dollars to distribute to n boxes.

4. What is the number of poker hands that have at least 1 ace? (Recall that a poker hand is 5 cards from a 52 card deck.)
   Answer: $\binom{52}{5} - \binom{48}{5}$. The latter term is the number of poker hands without an ace.

8. Probability Pts: 3/3/3/2/2/2/2/4/2/2/2/2

1. Consider rolling two six sided fair dice.

   (a) What is the probability that exactly one die is 6?
   Answer: $10/36$. There are five ways where the first die is 6 and the second is not, and five more vice versa.

   (b) What is the probability that the sum of the two dice is 6?
   Answer: $5/36$. The first die, could be $1, \ldots, 5$, and the other is determined for each first one.

   (c) What is the probability that the sum is 6 given that at least one die is at least 3?
   Answer: $5/32$. There are 4 ways where both die are less than 3, and there are 5 ways where the sum of the die is 6.

   (d) The event of rolling a 6 on the first die is independent of the event that the dice sum to 7.
   Answer: True. 1 out of the six ways to get a sum of 7 is the same is 6/36 ways overall.

   (e) The event of rolling at least one 6 is independent of the event that the dice sum to 7.
   Answer: False. The probability of at least one 6 is $11/36$ and the probability of getting at least one 6 when the die sum to 7 is $1/3$.

2. Flip a coin until you repeat either heads or tails 2020 times. We will derive the probability that the first coin is the same as the last coin in the entire sequence of flips.

   (a) If the process stops after 2020 tosses, what is the probability that the first and last coin are the same?
   Answer: 1

   (b) If the process stops after 2021 tosses, what it the probability that the first and last coin are the same?
   Answer: 0

   (c) What is the probability that the first coin is the same as the last coin in the entire sequence of flips?
   Answer: $1/2 + 2^{-2021}$. It $1 \times 2 \times 1/(2)^{2020} + 0 \times 2 \times (1/2)^{-2021} + (1/2)2(1 - 2 \times (1/2)^{2020} - 2 \times (1/2)^{2021})$

3. Which of the following are always true.

   (a) $E[10X] = 10E[X]$
   Answer: True. Linearity of expectation is $E[aX + b] = aE[X] + b$.

   (b) $E[X^2] = E[X]^2$
   Answer: False. This is not true in general.

(c) $E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[X]E[Y]$

Answer: False. Can't distribute linearity of expectation over term $E[XY]$.

(d) $Var(X+Y) = Var(X) + Var(Y)$

Answer: False. This is not always true, e.g., if $Y = X$, $Var(2X) = 4Var(X) \neq 2Var(X)$

9. Marbles: Pts: 4/4

Consider two bags of marbles, the "majority red" bag has 6 red marbles and 4 blue marbles, and the "majority blue" bag has 3 red marbles and 7 blue marble, and each bag is chosen with probability $1/2$.

1. If you draw a blue marble where each marble in the bag is equally likely, what is the probability that the bag is the "majority blue" bag.

   Answer: $7/11$. A - event it is majority blue. B is get a blue marble. Notice that each of the marbles is equally likely to be chosen since each bag is chosen with equal probability.

   Thus $Pr[B]$ is 11/20 since there are 11 blue marbles out of 20 marbles. And $Pr[A \cap B]$ is 7/20 as there are 7 blue marbles in the "majority blue" bag out of 20.

   Thus, $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = 7/11$.

2. What is the probability that the next marble is blue?

   Answer: Use total probability (or thinking about cases!) Let N be the event 'next blue'.
   $Pr[N] = Pr[N|A]Pr[A] + Pr[N|\overline{A}]Pr[\overline{A}] = \frac{6}{9}\frac{7}{11} + \frac{3}{9}\frac{4}{11} = 54/99 = \frac{6}{11}$.

10. Variance, covariance, tail bounds. Pts: 3/3/3/3/3

1. If $E[X] = 4$, and $E[Y] = 5$, and $E[XY] = E[X]E[Y]$, what is cov(X,Y)?

   Answer: 0. $cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - 2E[X]E[Y] + E[X]E[Y] = E[XY] - E[X]E[Y] = 0$.

2. A student earns one standard deviation above the mean on both exam 1 and exam 2. We define random variables X and Y as the score of a randomly chosen student on exam 1 and exam 2 respectively. If $Var(X) = 1$, $Var(Y) = 1$ and $Cov(X,Y) = .5$, how many standard deviations above the mean did the student get on the sum of her two scores? (Recall, $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.)

   Answer: $2/\sqrt{3}$. The student value is 2 over the sum of the means, and the variance is 3 of the sum of the random variables, so the standard deviation is $2/\sqrt{3}$.

3. For a random variable, X, where $X \geq -1$, $E[X] = 5$, and $E[X^2] = 26$. Give an upper bound $Pr[X \geq 6]$. (It should be tight with respect to the appropriate inequality.)

   Answer: $6/7$. One can use Markov's inequality on the random variable $X + 1$.

4. For a random variable, X, where $E[X] = 5$, and $E[X^2] = 26$, give an upper bound $Pr[X \geq 9]$. (It should be tight with respect to the appropriate inequality.)

   Answer: $\frac{1}{16}$.

   Chebyshev gives $Pr[(X - E(X)) \geq 4] \leq Pr[|X - E[X]| \geq 4] \leq \frac{E[X^2] - E[X]^2}{16} = \frac{1}{16}$.

5. Let X be $E[X] = 10$ and $Var[X] = \sigma^2$. Let $Y = \frac{X_1 + \cdots + X_n}{n}$ where X_i are i.i.d samples of X, for what value of n is $Pr[|Y - E[X]| \geq 1.0] \leq .05$? (Provide a bound that is as tight as possible using Chebyshev's inequality.)

   Answer: $n \geq 20\sigma^2$. The variance of Y is σ^2/n, and thus $Pr[|Y - E[X]| \geq 1.0] \leq \frac{Var(Y)}{1.0} = \frac{\sigma^2}{1.0n} \leq .05$
   Thus, $n \geq \frac{\sigma^2}{0.05} = 20\sigma^2$.

11. **Continuous: warmup. Pts: 2/2/2/3/3**

    Consider a continuous random variable, X, with pdf $f(x)$. (Answers below are a number or possibly expressions that involve the random variable and $E[\cdot]$ or $Var[\cdot]$.)

    1. What is $\int_{-\infty}^{\infty} f(x)dx$?
       Answer: 1. By definition of a probability density function.

    2. $\int_{-\infty}^{\infty} xf(x)dx$?
       Answer: $E[X]$. By definition of expectation.

    3. $\int_{-\infty}^{\infty} x^2 f(x)dx$?
       Answer: $E[X^2]$. Using the $E[g(x)]$ for $g(x) = x^2$.

    4. Consider $Y = 2X$ where $X \sim Expo(\lambda)$, $Y \sim Expo(\lambda')$, what is λ'?
       Answer: $\lambda/2$. We stated it was exponential and the expectation must double, so the parameter halves.

    5. Recall that for choosing a uniform point in a unit square the pdf is $f(x,y) = 1$ in the unit square and zero elsewhere. What is the pdf for choosing a uniform point in a 2×2 square? (Answer need only state the pdf inside the 2×2 square as outside it is zero.)
       Answer: $\frac{1}{4}$. The density drops by the factor that the area grows by.

12. **Distributions: continuous and discrete. Pts: 3/3/3/3/3/3/1/3**

    1. Given $X,Y \sim Binomial(n,p)$ what is the variance of $X+Y$?
       Answer: $2np(1-p)$. $X+Y \sim Binomial(2n,p)$ if they are *independent.*. Technially the correct answer is $2np(1-p) + 2(E[XY] - (np)^2)$.

    2. What is $E[\min(X,Y,Z)]$ where $X,Y,Z \sim Geometric(p)$?
       Answer: $\frac{1}{1-(1-p)^3}$.
       Let $M = \min(X,Y,Z)$, $Pr[M \geq i] = Pr[X \geq i, Y \geq i, Z \geq i] = (1-p)^{3(i-1)}$.
       The tail sum for expectation gives $\sum_1^{\infty} r^{i-1}$ for $r = (1-p)^3$ which is $\frac{1}{1-r}$.

    3. What is $E[\min(X,Y,Z)]$ where $X,Y,Z \sim Expo(\lambda)$?
       Answer: $\frac{1}{3\lambda}$. Let $M = \min(Y,X,Z)$, $Pr[M > x] = Pr[X > x, Y > x, Z > x] = (e^{\lambda x})^3$.
       This tells us the distribution is $Expo(3\lambda)$.

    4. Let $Z \sim Expo(\lambda)$ and $Y = \lceil Z \rceil$ (where $\lceil x \rceil$ is the lowest integer of value at least x). Note that the variable $Y \sim Geometric(p)$. What is the value of p in terms of λ?
       Answer: $Y \sim G(e^{-\lambda}))$.
       $Pr[Z \in [i, i+1] | Z \geq i] = Pr[Z \in [0,1]] = 1 - e^{-\lambda}$ by the memoryless property.

    5. Let $Y \sim Expo(\lambda)$, what is the conditional probability density function of Y if $Y \in [i, i+1]$ for a natural number i in the range $[i, i+1]$?
       [A] $\frac{\lambda e^{-\lambda(x)}}{e^{-\lambda i}}$ [B] $\frac{\lambda e^{-\lambda(x)}}{(1-e^{-\lambda})}$ [C] $\frac{\lambda e^{-\lambda(x-i)}}{(1-e^{-\lambda})}$ [D] $\frac{\lambda e^{-\lambda(x-i)}}{e^{-\lambda}(1-e^{-\lambda})}$ [E] $\frac{\lambda e^{-\lambda(x-i)}}{(1-e^{-\lambda})^i}$
       Answer: $\frac{\lambda e^{-\lambda(x-i)}}{(1-e^{-\lambda})}$.
       An simple way to see this is the memoryless property: it is the same as the conditional distribution of being in $[0,1]$ but the variable needs to be shifted into that range. This gives $e^{-\lambda y}/(1 - e^{-\lambda})$ and then use $y = (x-i)$.
       Alternative solution. One can also do the calculuation as follows.
       $Pr[Y \in [i, i+1]] = Pr[Y \leq i+1] - Pr[Y \leq i] = (1 - e^{-\lambda(i+1)}) - (1 - e^{-\lambda(i)}) = e^{-\lambda(i)}(1 - e^{-\lambda})$

and

$Pr[Y \in [x, x+dx]|Y \in [i, i+1]] \approx \frac{\lambda e^{\lambda(x)}dx}{e^{\lambda(i)}(1-e^{-\lambda})}$ if $x \in [i, i+1]$.

And we remove the dx to get the density.

6. For $X \sim Geometric(p)$ and $Y \sim Poisson(X)$, what is $E[Y]$?

   Answer: $\frac{1}{p}$. $\sum_x (\sum_y y Pr[Y = y])(1-p)^{x-1}p = \sum_x (\sum_y y Pr[Y = y])(1-p)^{x-1}p = \sum_i i(1-p)^{i-1}p = \frac{1}{p}$

7. Consider a random variable $X = 2\ln Y$ where $Y \sim U[0, 1]$.

   (a) What is the range of X? (The range is where the pdf of X is positive.)
   **Answer:** $[-\infty, 0]$. For $\ln x$ has range $[-\infty, 0]$ for $x \in [0, 1]$

   (b) What is the pdf of X on the range defined above? (Hint: $Pr[X \in [x, x+dx]] = Pr[Y \in [e^{x/2}, e^{(x+dx)/2}]$
   and $e^{x+dx} \approx e^x(1+dx)$)
   Answer: $\frac{e^{x/2}}{2}$. $Pr[Y \in [e^{x/2}, (1+dx/2)e^{x/2}]] \approx \frac{1}{2}e^{x/2}dx$ since $Y \sim U[0, 1]$.
   Thus, the pdf for X is $e^{x/2}/2$. One can confirm that $\int_{-\infty}^{0} \frac{1}{2}e^{x/2}dx = 1$.

13. Continuous: Triangle. Pts: 2/3/3/2

Consider a right equilateral triangle of side lengths $1, 1$ and $\sqrt{2}$. Given a random point in the triangle, we define the random variable X as the distance from the hypotenuse as shown in the figure below.
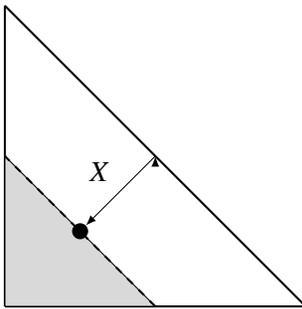
1. What is the joint density function $f(x, y)$ for points inside the triangle? (Again, the point is chosen uniformly inside the entire triangle. Ignore the shading in the figure for now.)

   Answer: 2. Note the area of the triangle is $1/2$, thus the density per unit area is 2.

2. What is the area of the shaded triangle in terms of X? (Hint: range of X is $[0, \sqrt{2}/2]$)

   Answer: $\frac{(1-\sqrt{2}x)^2}{2}$. The sidelength is $1 - x\sqrt{2}$.

3. What is the cdf of X for the range $x \in [0, \sqrt{2}/2]$?

   Answer: $1 - (1 - \sqrt{2}x)^2$.

   The points that are farther than x away from the hypotenuse are in the shaded triangle. The density is 2, which we multiply be the area of the shaded.
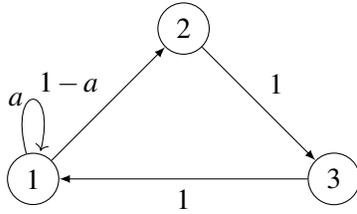
   You take the complement to get the cdf.

4. What is the pdf of X for the range $x \in [0, \sqrt{2}/2]$?

   Answer: $2\sqrt{2}(1 - \sqrt{2}x)$. Take the derivative of the cdf.

14. Markov Chain. Pts: 2/2/2/2

Consider the following Markov chain.

1. For what value of a does the chain have a unique invariant distribution but does not always converge to it.

   Answer: $a = 0.0$. In this case there is one cycle of length 3, and the period is 3.

2. For $a = 1/2$, what is the stationary distribution?

   $\pi(1)$: $\pi(2)$: $\pi(3)$:

   Answer: $\pi(1) = 1/2, \pi(2) = \pi(3) = 1/4$

   Let $x = \pi(1)$, $\pi(3) = \pi(2) = (1-a)x$. And $x + 2(1-a)x = 1.0$, which means $x = 1.0/(3-2a)$.

Long Answers Starting From here.

15. **Small faces. Pts: 2/2/4**

Given a planar graph with minimum degree 3 with e edges, v vertices and f faces we will prove there is a face of length at most 5. (The length of a face is the number of edges along it.)

1. What is the sum of the face lengths, $\sum_{i=1}^{f} s_i$ where s_i is the size of face i, in terms of e?

   Answer: $2e$. Each face is adjacent to s_i edges, and each edge is adjacent to 2 faces.

2. Give a lower bound on e in terms of v. (Hint: the minimum degree is 3.)

   Answer: Recall $2e = \sum_v d(v) \geq 3v$, or $e \geq \frac{3}{2}v$.

3. Prove that there is a face of size at most 5. (Recall: Euler's formula $v + f = e + 2$.)

   Answer: Assume that every face has length at least 6.

   $2e = \sum_{i=0} s_i \geq 6f$. or $\frac{1}{3}e \geq f$

   And

   $v + \frac{1}{3}e \geq e + 2$ or $v \geq \frac{2}{3}e + 2 \geq v + 2$. This is a contradiction.

16. **Balls and Bins.Pts: 3/3/2/2**

Consider placing $5n$ balls into $3n$ bins uniformly at random. (Careful, the constants in front of the n's are important.)

1. What is the expected number of empty bins?

   Answer: $3n(1 - \frac{1}{3n})^{5n}$. The probability that a bin is empty is $(1 - \frac{1}{3n})^{5n}$ and then use linearity of expectation.

2. What is the variance of the number of empty bins?

   Answer: $3n(1 - \frac{1}{3n})^{5n} + 2\binom{3n}{2}(1 - \frac{2}{3n})^{5n} - (3n(1 - \frac{1}{3n})^{5n})^2$

   Let X_i be an indicator random variable for the ith bin being empty.

   $E[(X_1 + \cdots X_{3n})^2] = \sum_i X_i^2 + \sum_{ij} X_i X_j = 3n(1 - \frac{1}{3n})^{5n} + 2\binom{3n}{2}(1 - \frac{2}{3n})^{5n}$

3. What is the expected number of non-empty bins?

   Answer: $3n(1 - (1 - \frac{1}{3n})^{5n})$. This is just $3n - X$ where X is the number of empty bins.

4. What is the variance of the number of non-empty bins (in terms of the answer for part (1),(2) and/or (3).)

Answer: Same as *b*. The $Var(3n - X) = Var(X)$.

17. Sequential Dice.Pts: 3/3/3

Consider rolling a dice repeatedly and until one gets two 6's in a row.

1. Draw a three state Markov chain where the states are labelled *A*,*B*, and *C*. Your chain should have a state *C* which is the "goal"; the previous two rolls were a 6. State *A* should indicate that one has not rolled any die or that the previous die is not 6.

Answer:
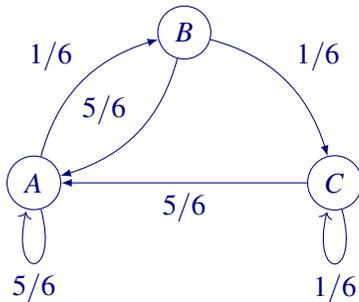
2. What is the expected number of rolls to roll two 6's in a row?

Answer: 42.

$\beta(a) = 1 + \frac{1}{6}\beta(b) + \frac{5}{6}\beta(a)$

$\beta(b) = 1 + \frac{5}{6}\beta(a)$

Thus,

$\beta(a) = 1 + \frac{1}{6}(1 + \frac{5}{6}\beta(a)) + \frac{5}{6}\beta(a)$

And, $\beta(a) = 42$.

3. What is the probability of rolling two 6's in a row prior to rolling a 5? (Hint: add a state to your previous Markov Chain and do a computation.)

Answer: $1/8$.

Add a state *D* with transition probabitiy from *A* and *B* of $1/6$.

$\alpha(a) = 2/3\alpha(a) + 1/6\alpha(b)$

$\alpha(b) = 2/3\alpha(a) + 1/6$

Thus, $\alpha(a) = 7/9\alpha(a) + 1/36$ and $\alpha(a) = 1/8$.

18. Bayes Rule. Pts: 5

A doctor has information that 80% of the sick children in a neighborhood *have the flu* and the other 20% of sick children have measles. He further knows that the probability of a rash with measles, is 0.95, and that the probability of a rash with flu is .10. If a sick child has a rash, what is the probability the child has measles. (Show your work here. And use the box for your final answer.)

Answer: $19/27$.

$$Pr[M|R] = \frac{Pr[R|M] \times Pr[M]}{Pr[R|M] \times Pr[M] + Pr[F|R] \times Pr[R]} = \frac{.2 * .95}{.2 + .10 \times .8} = \frac{19}{27}.$$

19. **Close enough! Pts: 5**

Given a circle (dartboard) of radius 1, choose two points at random on the dartboard uniformly, and let X and Y define the distance to the center. What is the probability that $|X - Y| \leq \delta$? (Recall that the pdf of both variables is $f(x) = 2x1\{x < 1\}$)

Answer: $\frac{4(1-\delta)^3\delta}{3} - 2\delta(1-\delta)^2$.

We compute the probability that $X \leq Y$ and $|Y - X| < \delta$.

$\int_0^{1-\delta} 2x \int_x^{x+\delta} 2y\,dy\,dx + \int_{1-\delta}^1 2x \int_x^1 2y\,dy\,dx$

$= \int_0^{1-\delta} 2x((x+\delta)^2 - x)dx + \int_{1-\delta}^1 (1 - x^2)dx$

$= \int_0^{1-\delta} 2x(2\delta x + \delta^2) + (x - \frac{x^3}{3})|_{1-\delta}^1$

$= \frac{2\delta x^3}{3} - \delta x^2|_0^{1-\delta} + (\delta - \frac{1}{3} + \frac{(1-\delta)^3}{3}$

$= \frac{2\delta(1-\delta)^3}{3} - \delta(1-\delta)^2 + \delta - \frac{1}{3} + \frac{(1-\delta)^3}{3}$

$= \delta(1-\delta)^3 - \delta(1-\delta)^2 + \delta - \frac{1}{3}$

FIXME...

$2\left(\frac{\delta^4}{6} - \delta^2 + \frac{4\delta}{3}\right)$

We double since the $Y \leq X$ case is symmetric.

20. **Puzzler: Pts: 3/5**

Consider the following game on an $n \times m$ grid, with two cooperating players. A key is hidden under a grid square and on each square there is a single coin that is either heads or tails. Player 1 knows the key location and *must flip exactly one coin.*

Player 2 should observe the pattern of heads and tails and produce the key location.

To reiterate, from an arbitrary initial setup of heads and tails on the grid, player 1 should flip exactly one coin to make a setup where player 2 can determine the location of the hidden key.

1. What is a strategy for the players to win on a 2×1 grid? (Hint: Think about $2x + y \pmod 2$ for $x, y \in \{0, 1\}$ and think of heads as 1 and 0 as tails.)

   Answer: Let's $x = 1$ if the left square is heads, and $y = 1$ if the right one is. Consider the $z = 2x + y \pmod 2$. The players agree that if $z = 1$, the key is in the left square, otherwise it is in the right one. Notice if the value of z in the initial configuration is correct changing the value of x does not change the value of z, so Player 1 can flip the coin in the first square, if initial value of z was incorrect, Player 1 can flip the coin in the right square and change it.

2. What is a strategy for the players to win on a $2^k \times 2^k$ grid? (Hint: use induction to find the column and row of the coin to flip. Notice, $2^k = (2 \times (2 \times 2^{k-1})).$)

   Answer:

   Break up the grid into 4 subgrids of size $2^{k-1} \times 2^{k-1}$.

   Then one can use a 2×2 grid using part (a) to figure out the column.

   One takes the parity of the number of heads in each subgrid to be a proxy for flipping one coin, as flipping *any coin* in that subgrid flips the parity.

   Now the 2×2 grid can be done, by taking parity of columns and using part (a) to determine the column and then to recursively choose the bit in the column to determine the row.

Thus, at the top level one has a 2×2 instance.

Then recurse on the quadrant that is indicated by the 2×2 instance which is now a $2^k \times 2^k$.

Another solution is to consider reorganize into a $2 \times 2^{2k-1}$ grid, and taking the parity of the columns to indicate the column, and then using that column recursively using the 2×1 solution in part (a).